Lorentz Transformation of Force

Lorentz force equation in covariant form is

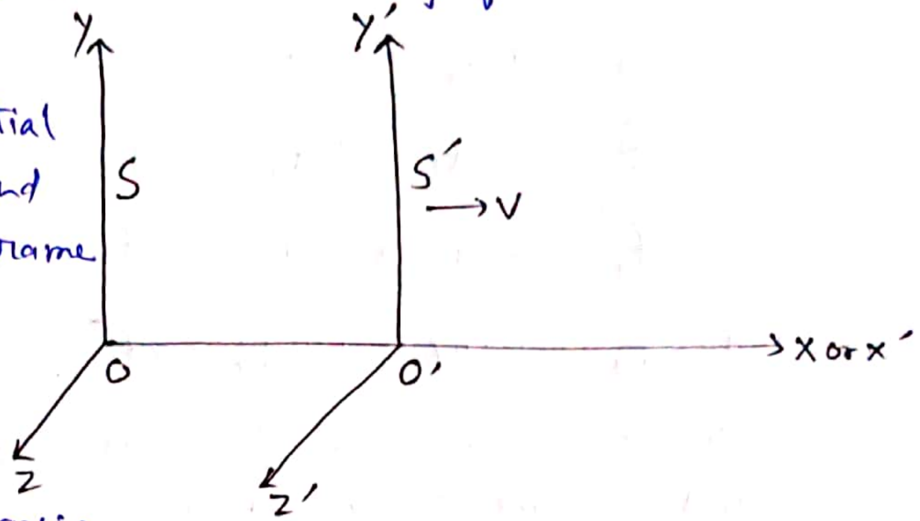
$$f_\mu = F_{\mu\nu} \cdot J_\nu \quad \text{——— (1)}$$

where $F_{\mu\nu}$ = Electromagnetic field tensor

f_μ = Force-density four vector

and J_ν = current density four vector.

In shown figure, there are two inertial refrence frames S and S′ in which the frame S is at rest and the frame S′ is moving with velocity v along positive x-axis relative to the frame S.



Lorentz Transformation equation of the four vector f_μ is

$$f'_\mu = \alpha_{\mu\nu} f_\nu \quad \text{——— (2)}$$

where f'_μ = Force density four vector as seen from S′ frame.

f_ν = Force density four vector as seen from rest frame S.

and transformation matrix $\alpha_{\mu\nu}$ is

$$\alpha_{\mu\nu} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

and $\beta = \frac{v}{c}$.

Using this transformation matrix $\alpha_{\mu\nu}$, the eqn (2) will become

$$\begin{pmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \times \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$$

On solving the equation, we get

$$\left.\begin{aligned} f_1' &= \gamma \cdot f_1 + 0 \cdot f_2 + 0 \cdot f_3 + i\beta\gamma \cdot f_4 \Rightarrow f_1' = \gamma \cdot f_1 + i\beta\gamma \cdot f_4 \\ f_2' &= 0 \cdot f_1 + 1 \cdot f_2 + 0 \cdot f_3 + 0 \cdot f_4 \Rightarrow f_2' = f_2 \\ f_3' &= 0 \cdot f_1 + 0 \cdot f_2 + 1 \cdot f_3 + 0 \cdot f_4 \Rightarrow f_3' = f_3 \\ \text{and } f_4' &= -i\beta\gamma \cdot f_1 + 0 \cdot f_2 + 0 \cdot f_3 + \gamma \cdot f_4 \Rightarrow f_4' = -i\beta\gamma f_1 + \gamma f_4 \end{aligned}\right\} \text{—(3)}$$

Since no work is done in the rest system of charges and frame S is at rest

so $f_4 = \frac{i}{c}(\vec{E} \cdot \vec{J}) = 0$

Using this, eqn (3) will become

$$\left.\begin{aligned} f_1' &= \gamma \cdot f_1 \\ f_2' &= f_2 \\ f_3' &= f_3 \end{aligned}\right\} \text{——— (4)}$$

$f_4' = -i\beta\gamma \cdot f_1$ It is left due to imaginary

We are considering only real forces so first three parts are taken.

Eqn (4) represents Lorentz transformation of force-density four vector.

The component of the total force exerted on a given volume of the charge distribution is

$\to d\tau'$ small volume in frame S'.

$$F_\alpha' = \int_\tau f_\alpha' \cdot d\tau'$$

The x-component of the total force exerted on a given volume of the charge distribution will be

$$F_x' = F_1' = \int_\tau f_1' d\tau' = \int \gamma \cdot f_1 \cdot \frac{d\tau}{\gamma} \quad \because f_1' = \gamma f_1, d\tau' = \frac{d\tau}{\gamma}$$

$$\Rightarrow F_x' = \int_\tau f_1 \cdot d\tau = F_1 = F_x$$

$$\Rightarrow \boxed{F_x' = F_x} \text{ --- (A)}$$

Again y-component is $F_y' = F_2' = \int_\tau f_2' \cdot d\tau' = \int_\tau f_2 \cdot \frac{d\tau}{\gamma}$ $\left(\because f_2' = f_2, \; d\tau' = \frac{d\tau}{\gamma}\right)$

$$\Rightarrow F_y' = \frac{1}{\gamma} \cdot \int_\tau f_2 \cdot d\tau = \frac{1}{\gamma} \cdot F_2 = \frac{1}{\gamma} \cdot F_y$$

$$\Rightarrow \boxed{F_y' = \frac{1}{\gamma} \cdot F_y = \sqrt{1 - \frac{v^2}{c^2}} \cdot F_y} \text{ --- (B)}$$

Again z-component is $F_z' = F_3' = \int_\tau f_3' \cdot d\tau' = \int_\tau f_3 \cdot \frac{d\tau}{\gamma}$ $\left(\because f_3' = f_3, \; d\tau' = \frac{d\tau}{\gamma}\right)$

$$\Rightarrow F_z' = \frac{1}{\gamma} \cdot \int_\tau f_3 \, d\tau = \frac{1}{\gamma} \cdot F_3 = \frac{1}{\gamma} \cdot F_z$$

$$\Rightarrow \boxed{F_z' = \frac{1}{\gamma} \cdot F_z = \sqrt{1 - \frac{v^2}{c^2}} \cdot F_z} \text{ --- (C)}$$

Eqn's (A), (B) and (C) can be written as

$F_{\parallel}' = F_{\parallel}$ → component in the direction of motion

and $F_\perp' = \frac{1}{\gamma} \cdot F_\perp = \sqrt{1 - \frac{v^2}{c^2}} \cdot F_\perp$ → component perpendicular to direction of motion.

These are general transformation (Lorentz transformation) for force either mechanical or electrical.

From general transformation for force, it is clear that component of force does not change in the direction of motion by changing frame from S to S′ but component of force change by a factor of $\frac{1}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}}$ in the direction perpendicular to direction of motion by changing frame from S to S′.

---x---

Scanned with CamScan